# SEMI-CLASSICAL STABILITY OF SUPERGRAVITY VACUA

Marika M. Taylor-Robinson
DAMTP
Silver Street
Cambridge
CB3 9EW

February 1, 2008

**Abstract**

We discuss the existence of instantonic decay modes which would indicate a semi-classical instability of the vacua of ten and eleven dimensional supergravity theories. Decay modes whose spin structures are incompatible with those of supersymmetric vacua have previously been constructed, and we present generalisations including those involving non trivial dilaton and antisymmetric tensor fields. We then show that the requirement that any instanton describing supersymmetric vacuum decay should admit both a zero momentum hypersurface from which we describe the subsequent Lorentzian evolution and a spin structure at infinity compatible with the putative vacuum excludes all such decay modes, except those with unphysical energy momentum tensors which violate the dominant energy condition.

# 1 Introduction

Supergravity theories exist in all spacetime dimensions $d$ with $d \leq 11$, and are currently regarded as effective field theories of superstring (and M) theories in some appropriate limit. Classical solutions of the theories can be found by setting to zero the fermionic fields together with their supersymmetric variations. We look for a vacuum in which the space-time is of the form $B^4 \times K$ where $B^4$ is a maximally symmetric four-dimensional space (de Sitter space, anti de Sitter space or Minkowski space) and K is a compact manifold; such a solution is consistent with the low-energy field equations, with the dilaton field constant and all other fields vanishing. The conditions for finding supersymmetric generators that leave the vacuum invariant restrict $B^4$ to be flat Minkowski space and $K$ to be a manifold that admits at least one covariantly constant spinor field. This in turn constrains the possible holonomy groups of $K$; for ten dimensional theories, $K$ must have a holonomy contained in $SU(3)$ [1], implying that $K$ must have a covering space that is $T^6$, $T^2 \times K3$ or a Calabi-Yau space $K_{SU(3)}$. Similarly, for eleven dimensional supergravity, the holonomy of $K$ is contained in $Spin(7)$, and $K$ has a covering space that is $T^7$, $T^3 \times K3$, $K_{SU(3)} \times S^1$ or $K_{Spin(7)}$ (where the latter is a manifold with the exceptional holonomy group $Spin(7)$) [2].

It is important to have some criteria for determining whether $M^4 \times K$ is a reasonable candidate as the ground state of supergravity theories, but our incomplete understanding of string (and M) theory dynamics makes this question difficult to answer in full. The constraints above restrict the vacuum state to be Ricci-flat, with the requisite holonomy; we must, however, also show that the spectrum of the vacuum is stable and that there are no instantonic decay modes, ie. we must thus impose the conditions that $M^4 \times K$ should be stable at the classical and semi-classical level, which leads to non-trivial conditions on the vacuum manifold.

The first test of the stability of a space is to ask whether the space is stable classically against small oscillations. Small oscillations around $M^4 \times K$ will consist of a spectrum of massless states (the graviton, gauge fields, dilaton etc) and an infinite number of charged massive modes. The massless spectrum of the heterotic string theory, which is the theory that we will consider principally here, has been extensively discussed (see for example [1], [3], [4] and [2]); there are no exponentially growing modes with imaginary frequencies. The same applies to that of eleven dimensional supergravity.

Even if a state is stable against small oscillations, it may be unstable at the semiclassical level. This can occur if it is separated by only a finite barrier from

a more stable state; it will then be unstable against decay by semiclassical barrier penetration. To look for a semiclassical instability of a putative vacuum state, one looks for a bounce solution of the classical Euclidean field equations; this is a solution which asymptotically at infinity approaches the putative vacuum state. If the solution is unstable, then the Gaussian integral around that solution gives an imaginary part to the energy of the vacuum state, indicating an instability.

The stability of Minkowski space at the semi-classical level as the unique vacuum state of general relativity was proved by the positive energy theorem of Schoen and Yau [5]. A completely different proof involving spinors satisfying a Dirac type equation on a three-dimensional initial value hypersurface was given by Witten [6], and shortly after re-expressed in terms of the Nester tensor [7]. Witten later demonstrated the instability of the $M^4 \times S^1$ vacuum of Kaluza-Klein theory [8]; the effective four-dimensional vacuum decays into an expanding bubble of "nothing". However, this decay mode is excluded by the existence of (massless) elementary fermions.

Instabilities of non-supersymmetric vacua of string theories were discussed by Brill and Horowitz [9] who demonstrated that superstring theories admit instantonic decay modes that asymptotically resemble toroidal compactifications with constant gauge fields (but are incompatible with massless fermions). Mazur [10] showed that toroidal compactifications of multidimensional Minkowski space-time are semiclassically unstable due to topology change of the initial data hypersurface, and presented Euclidean Schwarzschild p-branes as possible instantons corresponding to tunnelling between different topologies. However, the instantons discussed by Mazur do not correspond to vacuum instabilities since they do not take account of the incompatible spinor structures of the instantons and (supersymmetric) vacua. Banks and Dixon [11] used conformal field theory arguments to show that spacetime supersymmetry cannot be continuously broken within a family of classical vacua and that two supersymmetric vacua are infinitely far away. It was suggested in [12] that if one takes into account target space duality all topology changing instabilities of toroidal vacua are impossible in the context of string theory.

More recently, another possible decay mode of the Kaluza-Klein vacua has been constructed by Dowker et al. [13], [14]. "Magnetic" vacua in Kaluza-Klein theory - vacua corresponding to static magnetic flux tubes in four dimensions - may decay by pair creation of Kaluza-Klein monopoles, though at a much smaller rate than for decay by bubble formation; the pair creation decay mode is however consistent with the existence of elementary fermions.

In this paper, we investigate further possible decay modes of the vacua of supergravity theories. Such instantons certainly do not preserve the supersymmetry of the vacuum; however, we cannot *a priori* exclude the possibility of the vacuum decaying into a state which asymptotically admits the supersymmetry generators of the vacuum. That is, there may exist solutions to the Euclidean field equations both whose geometry is asymptotic to that of the background vacuum state, and whose spin structure at infinity is compatible with that of the supersymmetric vacuum.

In §2 we describe the vacua of ten-dimensional supergravity theories, and discuss new examples of twisted compactifications which give rise to magnetic vacua in four dimensions. In the following section, we discuss Ricci-flat instantons which describe decay of toroidal vacua by bubble formation and pair creation of monopoles. In §4 we consider more general instantons, relaxing the assumption of Ricci flatness, and construct from five-dimensional charged black hole solutions decay modes involving non-vanishing dilaton and antisymmetric tensor fields.

In §5 we use extremal black hole solutions with non-degenerate horizons to describe decay modes whose topology is not inconsistent with asymptotically covariantly constant spinors, but whose energy momentum tensors are unphysical, violating the dominant energy condition.

In §6 we discuss more generally the existence of instantons describing the decay modes of the supersymmetric vacuum; we consider the formulation of Witten's proof of the positive energy theorem, and show how this proof excludes the existence of physical decay modes of a supersymmetric vacuum. In §7 and §8 we extend the discussion to the Calabi-Yau vacuum of ten dimensional supergravity and to the vacua of eleven-dimensional supergravity. Finally, in §9 we present our conclusions.

Note that contrary to globally supersymmetric Yang-Mills theories, supergravity is not renormalisable. This puts the entire subject of instanton calculus in supergravity on a rather shaky basis; if however we regard supergravity theories as low energy limits of superstring theories, which are not expected to suffer from these deficiencies, to the order to which supergravity theories are formally renormalisable results from non-perturbative instanton calculations may be considered as the limiting values of the corresponding exact string theory results [15].

Since manifolds of many different dimensions will abound, we will adhere to the following conventions. The indices $M, N = 0, .., 9$; $m, n = 1, .., 9$; $\mu, \nu = 0, .., 3$; $i, j = 4, .., 9$; $\alpha, \beta = 1, .., 3$; $a, b = 0, ..4$; $I, J = 5, .., 9$; $A, B = 0, .., 5$, $F, G = 6, .., 9$, $w, x = 0, 10$ and $f = 1, .., 16$. We use the mostly positive convention for Lorentzian metrics and $G$ will denote metrics in the string frame whilst $g$ denotes metrics in the

Einstein frame. $c$ denotes induced metrics on boundaries at spatial infinity, whilst $\hat{g}$ denotes induced metrics on spacelike hypersurfaces. Hatted indices refer to an orthonormal frame whilst unhatted indices refer to coordinate indices. $S$ denotes the Lorentzian action and $S_E$ denotes the Euclidean action. $G_d$ refers to the Newton constant in $d$ dimensions.

# 2 Vacua of supergravity theories

Our starting point is the $d = 10, N = 1$ action that arises as a low energy effective field theory from the heterotic string. We shall consider heterotic string theory for definiteness but most of the discussion depends only on the common sector of the low energy supergravity theory. For the massless bosonic fields of the theory (graviton $G_{MN}$, dilaton $\Phi$, antisymmetric tensor $B_{MN}$, 16 vector bosons $A^f_M$) the action (in the string frame) takes the form:

$$S = \frac{1}{16\pi G_{10}} \int d^{10}x\sqrt{-G}e^{-\Phi}\{R(G) + (\partial\Phi)^2 - \frac{1}{12}H^2 - \frac{1}{4}\mathrm{Tr}(\mathrm{F}^2)\} \qquad (2.1)$$

As in general relativity, this action will give the correct field equations but in calculating the action we must also include the surface terms required to ensure unitarity. The corresponding action in the Einstein frame $g_{MN} = e^{-\frac{\Phi}{4}}G_{MN}$ is:

$$S = \frac{1}{16\pi G_{10}} \int d^{10}x\sqrt{-g}\{R - \frac{1}{8}(\partial\Phi)^2 - \frac{1}{12}e^{-\frac{\Phi}{2}}H^2 - \frac{1}{4}e^{-\frac{\Phi}{4}}Tr(F^2)\} \qquad (2.2)$$

To find classical solutions of the supersymmetric theory, we set to zero the fermion fields (gravitino $\psi_M$, dilatino $\lambda$, gaugino $\chi$) together with their supersymmetric variations. Assuming that all fields vanish except the graviton (and constant dilaton) the conditions for finding a supersymmetric generator $\eta$ that leaves the vacuum invariant reduce to:

$$\delta\psi_M = D_M\eta = 0 \qquad (2.3)$$

As is well-known, and was first shown in [1] (see also [3]), for a vacuum state of the form $B^4 \times K$, where $B^4$ is a maximally symmetric four-dimensional space and $K$ is a compact six manifold, equation (2.3) implies that the maximally symmetric manifold must be flat Minkowski space, and $K$ must be a Ricci-flat compact six manifold, which admits at least one covariantly constant spinor of each chirality. The holonomy group of $K$ is thus constrained to be a subgroup of the generic

holonomy group of a six-dimensional manifold $SO(6)$, and hence the covering space of $K$ must be $T^6$, $T^2 \times K3$ or a Calabi-Yau space.

The simplest manifold which satisfies (2.3) and is Ricci-flat is the flat torus; we may also consider another class of toroidal vacua which asymptotically tend to static magnetic configurations in four dimensions. Although these solutions are non-trivial four dimensional configurations they are simply obtained from dimensional reduction of Euclidean space with twisted identifications. The construction of a static cylindrically symmetric flux tube in four dimensions by dimensional reduction of five dimensional Minkowski space in which points have been identified in a nonstandard way was discussed in [16], whilst more recently the construction was generalised to obtain sets of orthogonal fluxbranes in higher dimensional spacetimes [14].

We will now consider a four dimensional vacuum solution in which there are $p$ magnetic fields, arising from the ten-dimensional metric, associated with $p$ distinct $U(1)$ isometry groups. We start with 10-dimensional Minkowskian space and identify points under combined spatial translations and rotations, ie.

$$ds^2 = -dt^2 + dz^2 + d\varrho^2 + \varrho^2 d\psi^2 + \sum_i (dx^i)^2 \tag{2.4}$$

where we identify points $(x^i, t, z, \varrho, \psi) \sim (x^i + 2\pi\mu^i n^i, t, z, \varrho, \psi + \sum_i 2\pi B^i \mu^i n^i + 2\pi n)$ and $n, n^i$ are integers. We will usually assume that the $\mu^i$ are identical. Since $\psi$ is already periodic, changing $\sum_i \mu^i B^i$ by an integer does not change the identifications; thus inequivalent spacetimes are obtained only for $-1/2 < \sum_i \mu^i B^i \leq 1/2$. Changing each $B^i$ by a multiple of $1/\mu^i$ leads to equivalent spacetimes in $(4+p)$ dimensions, though the four dimensional configurations are not equivalent. Geometrically. this spacetime is obtained by starting with (2.4) and identifying points along the closed orbits of the Killing vectors $l^i = \partial_{x^i} + B^i \partial_\psi$. Introducing a new coordinate $\bar{\psi} = \psi + \sum_i B^i x^i$, we may rewrite the metric as

$$ds^2 = -dt^2 + dz^2 + d\varrho^2 + (dx^i)^2 + \varrho^2 (d\bar{\psi} + \sum_i B^i dx^i)^2 \tag{2.5}$$

Dimensionally reducing along the 6 Killing vectors $k^i = \partial_{x^i}$, the four-dimensional metric in the Einstein frame $g_{\mu\nu}$ is related to the 10-dimensional metric by:

$$g_{MN} = \begin{pmatrix} e^{\varphi} g_{\mu\nu} + \sum_{i,j} \xi_{ij} A^i_\mu A^j_\nu & \sum_i A^i_\mu \xi_{ij} \\ \sum_j A^j_\nu \xi_{ij} & \xi_{ij} \end{pmatrix} \tag{2.6}$$

with Kaluza Klein gauge fields $A^i$ and $\varphi$ the four-dimensional dilaton defined by $\varphi = \Phi - \frac{1}{2}\ln(\det\xi_{\rm ij})$. From (2.5), we find that:

$$\xi_{ii} = (1 + (B^i)^2 \varrho^2) \tag{2.7}$$

$$\xi_{ij} = B^i B^j \varrho^2 \tag{2.8}$$

$$A^i_{\bar\psi} = \frac{B^i \varrho^2 [1 - 2\sum_{j\neq i} B^j A^j_{\bar\psi}]}{(1 + (B^i)^2 \varrho^2)} \tag{2.9}$$

The off-diagonal terms in the internal metric imply that the torus is not a direct product of circles. The four-dimensional metric is given by

$$ds^2_{ein} = (1 + \sum_i (B^i)^2 \varrho^2)^{\frac{1}{2}} [-dt^2 + dz^2 + d\varrho^2] + g_{\bar\psi\bar\psi} d\bar\psi^2 \tag{2.10}$$

where $g_{\bar\psi\bar\psi}$ is defined by:

$$g_{\bar\psi\bar\psi} = (1 + \sum_i (B^i)^2 \varrho^2)^{\frac{1}{2}} [\varrho^2 - \sum_{i,j} \xi_{ij} A^i_{\bar\psi} A^j_{\bar\psi}] \tag{2.11}$$

The gauge fields in four dimensions are obtained by solving the $p$ simultaneous equations defined in (2.9); in the limit that only one field is non-zero, the solution reduces to the static magnetic flux tube. Asymptotically, each gauge field $A^i \rightarrow \frac{1}{pB^i}$; the gauge fields correspond to magnetic fields which are uniform at infinity.

We may also dimensionally reduce along the Killing vectors $\tilde{k}^i = k^i + (n^i/\mu^i)\partial_{\bar\psi}$; the corresponding four dimensional solution is unchanged, except that the magnetic field is modified to $\tilde{B}^i = B^i + n^i/\mu^i$, and in this way all values of the four-dimensional magnetic fields associated with each $U(1)$ can be obtained. For every $B^i \neq 0$, the proper length of the circles in the $i$th direction grows linearly with $\varrho$ for large $\varrho$; thus, we can view the solution as an approximation to physical fields which is valid only for $\varrho \ll 1/B^i$, in which range the three dimensional space is approximately flat, and the internal circles have approximately constant length. In order for the internal directions to remain unobservable, we must consider length scales which are large compared to their size: $\varrho \gg \mu^i$. The two restrictions imply a limited range of applicability of the spacetime, $B^i \ll 1/\mu^i$, which can include large magnetic fields only if the compactified dimensions are of the Planck scale. Since the different dimensional reductions change $B^i$ by multiples of $1/\mu^i$, for given $\mu^i$, at most one is physically reasonable.

These solutions can be obtained by the action of generating transformations on the original Kaluza-Klein solution of [16]; the required transformations are an $O(6)$ subgroup of the $O(6,22)$ T-duality group of the four-dimensional theory. The transformation acts as:

$$\begin{aligned} \tilde{A}^i_\mu &= \Omega_{ij} A^j_\mu \\ \tilde{\xi} &= \Omega^T \xi \Omega \end{aligned} \tag{2.12}$$

where $\Omega$ is an $O(6)$ invariant matrix, and all other fields are left invariant. The particular transformation required here is (assuming that the radii of the compactified directions are identical) $R_6(\vec{k})$, a six-dimensional rotation that rotates an arbitrary six-dimensional vector in direction $\vec{k}$ into a vector of the same magnitude with only one component non-zero.

Consider a solution for which $p$ fields are non-zero and equal to $B$:

$$ds^2_{ein} = e^{-\varphi}[-dt^2 + dz^2 + d\varrho^2] + \varrho^2 e^{\varphi} d\bar{\psi}^2$$

$$A^i_{\bar{\psi}} = \frac{B\varrho^2}{(1 + pB^2\varrho^2)} \tag{2.13}$$

$$e^{-2\varphi} = (1 + pB^2\varrho^2) \tag{2.14}$$

This has the interpretation of a flux tube along the z-axis, associated with $p$ magnetic fields, and is the required background for nucleation of monopoles carrying charges with respect to p $U(1)$ fields.

Even though these spacetimes are locally flat, the nontrivial identifications imply that if a vector is parallely transported around each $S^1$, it will return rotated by an angle $2\pi\mu^i B^i$. It follows that for one spin structure, parallel propagation of a spinor around the $i$th direction results in the spinor acquiring a phase $e^{\pi\mu^i B^i \gamma}$, where $\gamma$ is a generator of the Lie algebra of $SO(9,1)$ (spinor representation). For the other spin structure, parallel propagation gives a phase $-e^{\pi\mu^i B^i \gamma}$. For small $B^i$, the natural generalization of the standard choice of spinor structure for a supersymmetric vacuum is the first choice. The magnetic vacua evidently admit no covariantly constant spinors, whereas for the standard metric on the torus, 32 constant spinors are admitted.

Note that the invariance of the low energy effective action under the $O(6,22)$ T-duality group, and the invariance of the equations of motion under $SL(2,R)$ S-duality transformations allows us to generate further solutions. We apply a particular S-duality transformation $\varphi \to \tilde{\varphi} = -\varphi$ (corresponding to strong/weak coupling

interchange), $F^{i+6}_{\mu\nu} \to \tilde{F}^{i+6}_{\mu\nu} = e^{-2\varphi}\xi_{ij}\bar{F}^{j}_{\mu\nu}$ (with $\bar{F}$ the dual of $F$) and $F^{i}_{\mu\nu} \to \tilde{F}^{i}_{\mu\nu} = 0$. Then, rescaling to the string metric, $G_{\mu\nu} = e^{\tilde{\varphi}}g_{\mu\nu}$, the four-dimensional solution becomes:

$$\begin{aligned} ds^2_{str} &= e^{2\tilde{\varphi}}(-dt^2 + dz^2 + d\varrho^2) + \varrho^2 d\bar{\psi}^2 \\ \tilde{F}^{i+6}_{\tau z} &= \frac{2B}{(1 + B^2\varrho^2 p)^{\frac{1}{2}}} \qquad (2.15) \\ e^{2\tilde{\varphi}} &= (1 + B^2\varrho^2 p) \qquad (2.16) \end{aligned}$$

Thus, the only non-vanishing gauge fields are those originating from the off-diagonal components $B_{\mu i}$ of the two-form in ten dimensions. The solution describes an 'electric' flux tube, associated with $p$ gauge fields; each gauge field asymptotically approaches zero.

By applying a general $O(6, 22)$ generating transformation, we can obtain four-dimensional solutions describing tubes of magnetic flux associated with the $U(1)^{28}$ gauge group of the heterotic theory; these are the required backgrounds for nucleation of other topological defects, such as H-monopoles.

# 3 Ricci-flat instantons

We firstly consider instanton solutions of the Euclidean field equations in ten dimensions, in which all fields except the graviton and the (constant) dilaton vanish, implying that $R_{MN} = 0$. The asymptotic geometry of the instanton is $R^4 \times T^6$; we defer the discussion of Calabi-Yau vacua to §7. Evidently decay modes cannot preserve all the supersymmetry; 32 constant spinors requires trivial holonomy, implying that the solution admits a flat metric. However, if an instanton is to describe a possible vacuum decay mode, it must asymptotically admit the constant spinors of the background. Vacuum instability - which in many cases will correspond to physical formation processes - will hence result only from considering non-supersymmetric states which are not simple metric products, but rather contain topological defects such as monopoles or p-branes.

The instantons will usually globally admit a $U(1)^6$ isometry group, as well as a hypersurface orthogonal Killing vector which we use to Wick rotate the solution to describe the subsequent Lorentzian evolution. We may also of course consider instantons which admit such $U(1)$ isometries only asymptotically, although it is

unclear how the effective four-dimensional solution can be interpreted in this case. Fixed points of these isometries will lead to apparent singularities such as bubbles and monopoles in the lower-dimensional spacetime; the structure of these fixed point sets determines whether the instanton has a spin structure consistent with that of the background.

For a ten-dimensional instanton, the fixed point set must have dimension 10, 8, 6, 4, 2, 0; the classification of four-dimensional gravitational instantons in terms of the fixed points sets of a $U(1)$ isometry was discussed in [17] and reviewed in [18]. This work has been generalised to higher dimensions in [14] and [19].

If the isometry admits no fixed point sets, there is *a priori* no obstruction to choosing the spin structure of the instanton to be consistent with that of the background. If however the fixed point set of the isometry is eight-dimensional, spinors must be antiperiodic about a closed orbit of the isometry at infinity, and the spin structure is incompatible with that of the (supersymmetric) vacuum. The twisted boundary conditions break supersymmetry, and, although this supersymmetry breaking can be made arbitrarily weak by taking the compactified directions to be arbitrarily large, the action of the instanton diverges as the radii approach infinity, implying that the rate of decay of the vacuum goes to zero.

The obvious example is the five-dimensional Euclidean Schwarzschild solution crossed with a flat torus (a decay mode first considered in [6]):

$$ds^2 = dx^I dx_I + (1 - \frac{\mu}{r^2})d\tau^2 + \frac{dr^2}{(1 - \frac{\mu}{r^2})} + r^2 d\Omega_3^2 \tag{3.17}$$

where the periodicity of $\tau$ is $\Delta\tau = 2\pi\sqrt{\mu}$ and the range of $r$ is $r \geq \sqrt{\mu}$. Since the topology of the solution is $R^2 \times S^3 \times T^5$, the spin structure is incompatible with that of the supersymmetric vacuum. The action for this instanton is obtained from the boundary term:

$$S_E = -\frac{1}{16\pi G_{10}} \oint d^9x\sqrt{c}\{\mathcal{K} - \mathcal{K}_0\} \tag{3.18}$$

with $\mathcal{K}$ the trace of the second fundamental form of the boundary at infinity, and $\mathcal{K}_0$ the corresponding term in the background. The action is hence:

$$S_E = \frac{\pi\mu}{8G_4} \tag{3.19}$$

and thus the rate of vacuum decay does indeed vanish as the radii of the compactified directions increase, that is, as the supersymmetry breaking becomes arbitrarily weak.

Furthermore, the decay rate is small if $\sqrt{\mu} \gg$ Planck length, and it is only in this case that the semiclassical calculation is reliable.

If the fixed point set of the isometry has dimension less than eight, the spinors need not necessarily be antiperiodic about a closed orbit of the isometry at infinity. As an example, we may consider dimensional reduction of (3.17) along the Killing vector $k = \partial_\tau + \frac{1}{R}\partial_\psi$ [13] (where $R$ is the radius of the circle at infinity) which describes monopole pair creation in a background field. It is possible to choose the spin structure to be consistent at infinity with that of the solution describing the background field (2.5), but the magnitude of the magnetic field lies far outside the physical range of validity.

As also discussed in [13], we may consider instantons which are a product of $T^5$ and the five-dimensional Euclidean Kerr-Myers-Perry solution:

$$\begin{aligned} ds^2 &= (dx^F)^2 + dx^2 + dy^2 + \sin^2\theta(r^2 - \alpha^2)(d\psi)^2 + \frac{\rho^2}{r^2 - \alpha^2 - \mu}dr^2 \\ &\quad + \rho^2 d\theta^2 + r^2\cos^2\theta d\tau^2 - \frac{\mu}{\rho^2}(dx + \alpha\sin^2\theta(d\psi))^2 \end{aligned} \tag{3.20}$$

where the index $F$ runs over the coordinates of the remaining $T^4$ and $\rho^2 = r^2 - \alpha^2\cos^2\theta$. The most general such solution is labelled by one mass parameter and two angular momentum parameters (associated with the $(S0(2))^2 \times O(2)$ isometry group), but for simplicity we take only the mass parameter $\mu$ and one angular momentum parameter $\alpha$ to be non-zero.

Reduction along $\partial_x + \frac{\alpha}{\mu}\partial_\psi$, which has an eight-dimensional fixed point set, leads to decay of the four-dimensional vacuum by bubble formation, whilst reduction along $\partial_x + (\frac{\alpha}{\mu} + \frac{1}{R})\partial_\psi$, which has a six-dimensional fixed point set, leads to decay by monopole pair production. In the latter case, the four-dimensional field $B = \frac{\alpha}{\mu} + \frac{1}{R}$, and for $\frac{\alpha}{\mu} \sim -\frac{1}{R}$, we obtain fields of physical validity. The pair creation decay mode has a spin structure consistent with that of the magnetic vacuum and the action for the instanton (3.20) is:

$$S_E = \frac{\pi R^2}{8G_4(1 - R^2(\frac{\alpha}{\mu})^2)} \tag{3.21}$$

so for physical $B \ll 1/R$ the decay rate $\Gamma \sim e^{-S_E}$ is very small.

In §2, we showed that by applying a generating transformation to a solution in which there was a single non-zero magnetic field in four dimensions we could

obtain a solution in which there were several non-zero (metric) $U(1)$ fields in four dimensions. Using the same generating techniques here, we expect to obtain more general solutions describing the pair creation of monopoles carrying charge with respect to $p$ $U(1)$ gauge fields in a background of $q$ $U(1)$ gauge fields ($p, q \leq 6$). Monopoles carrying several different charges have recently been constructed [20] within heterotic string theory by the supersymmetric uplifting of four-dimensional monopole solutions; we now discuss their nucleation.

The most general solution Ricci-flat solutions will be obtained from the three parameter five dimensional Euclidean Kerr-Myers-Perry solution (constructed in [21]) crossed with a flat five torus by first applying an $O(6)$ transformation to $\xi$ and $A$, and then identifying points along closed orbits of $\partial_{x^i} + B^i\psi$ where the $B^i$ are chosen so that the action of the isometry is periodic.

We present an illustrative solution, describing pair creation of monopoles carrying two identical $U(1)$ charges within two (equal) background fields. We identify points along closed orbits of $\partial_x + B\partial_\psi$, and $\partial_y + B\partial_\psi$ in (3.20) and then introduce $\bar{\psi} = \psi - B(x+y)$. The required generating transformation is:

$$\Omega = \begin{pmatrix} R_2 & 0 \\ 0 & I_4 \end{pmatrix} \tag{3.22}$$

where $\Omega$ acts on $\xi$ and $A$ as in §2.1, and $R_2$ generates a two-dimensional transformation by $\pi/4$. The resulting solution is:

$$\begin{aligned} ds^2 \;=\;& (dx^F)^2 + (1 - \frac{\mu(1+\alpha\sqrt{2}B\sin^2\theta)^2)}{2\rho^2} + B^2(r^2-\alpha^2)\sin^2\theta)\mathrm{dx}^2 \\ & +2\,(-\frac{\mu(1+\alpha\sqrt{2}B\sin^2\theta)^2}{2\rho^2} + B^2(r^2-\alpha^2)\sin^2\theta)\mathrm{dxdy} \\ & +(1 - \frac{\mu(1+\alpha\sqrt{2}B\sin^2\theta)^2}{2\rho^2} + B^2(r^2-\alpha^2)\sin^2\theta)\mathrm{dy}^2 \\ & +(B^2(r^2-\alpha^2)\sin^2\theta\sqrt{2} - \frac{\mu\alpha\sin^2(\theta)(1+\alpha\sqrt{2}\mathrm{B}\sin^2\theta)\sqrt{2}}{2\rho^2})\mathrm{dxd}\bar{\psi} \\ & +(B^2(r^2-\alpha^2)\sin^2\theta\sqrt{2} - \frac{\mu\alpha\sin^2(\theta)(1+\alpha\sqrt{2}\mathrm{B}\sin^2\theta)\sqrt{2}}{2\rho^2})\mathrm{dyd}\bar{\psi} \\ & +\frac{\rho^2}{r^2-\alpha^2-\mu}dr^2 + \rho^2 d\theta^2 + r^2\cos^2\theta\mathrm{d}\tau^2 \end{aligned} \tag{3.23}$$

$$+ ((r^2 - \alpha^2)\mathrm{sin}^2\theta - \frac{\mu\alpha\mathrm{sin}^4\theta}{\rho^2})\mathrm{d}\bar{\psi}^2$$

where $\rho^2 = r^2 - \alpha^2\mathrm{cos}^2\theta$. The four-dimensional solution obtained by dimensionally reduced along closed orbits of $\partial_x$ and $\partial_y$ is:

$$\begin{aligned} ds_4^2 &= e^{-\varphi}\{r^2\mathrm{cos}^2\theta\mathrm{d}\tau^2 + \frac{\rho^2}{\mathrm{r}^2 - \alpha^2 - \mu}\mathrm{dr}^2 \\ &\quad +\rho^2 d\theta^2 + e^{2\varphi}(r^2 - \alpha^2 - \mu)\mathrm{sin}^2\theta)\mathrm{d}\bar{\psi}^2\} \\ e^{-2\varphi} &= 1 - \frac{\mu}{\rho^2}(1 + \alpha\sqrt{2}B\mathrm{sin}^2\theta)^2 + 2\mathrm{B}^2(\mathrm{r}^2 - \alpha^2)\mathrm{sin}^2\theta \\ A^x_{\bar{\psi}} &= \frac{\hat{g}_{yy}\hat{g}_{x\bar{\psi}} - \hat{g}_{xy}\hat{g}_{y\bar{\psi}}}{\xi} \\ A^y_{\bar{\psi}} &= \frac{\hat{g}_{xx}\hat{g}_{y\bar{\psi}} - \hat{g}_{xy}\hat{g}_{x\bar{\psi}}}{\xi} \end{aligned} \tag{3.24}$$

where $\xi = e^{-2\varphi}$ is the determinant of the metric on the torus; the ten-dimensional solution is complete and non-singular.

If we take $B = \alpha/\sqrt{2}\mu$, then the metric is singular over all the horizon $r_h^2 = \alpha^2 - \mu$; the solution describes a generalised bubble decay mode of the vacuum.

We may also take $B = \alpha/\sqrt{2}\mu + n/\sqrt{2}R$, in which case $e^{-2\varphi}$ vanishes only at the poles of the horizon; this is the solution describing pair creation of monopoles. The horizon is a line, which is smooth provided that we take $n = \pm 1$; for $|n| > 1$, the singularities at the poles are joined by a string. Since

$$R = \frac{\mu}{\sqrt{\alpha^2 + \mu}} < \frac{\mu}{\alpha} \tag{3.25}$$

to obtain four-dimensional magnetic fields of physical magnitude we need either $\alpha/\mu$ negative, close to $-1/R$ and $n = 1$ or $\alpha/\mu$ positive, close to $1/R$ and $n = -1$.

Consider the spin structure of the transformed solution. A spinor parallely transported about an orbit of $l_x = \partial_x + \alpha/\sqrt{2}\mu\partial_\psi$ can be shown to pick up a phase of $-e^{\pi R\sqrt{2}(\alpha/\sqrt{2}\mu)\gamma}$; the same phase is picked up by a spinor transported about an orbit of $l_y = \partial_y + \alpha/\sqrt{2}\mu\partial_\psi$. The four-dimensional magnetic fields are $B^x = B^y = \alpha/\sqrt{2}\mu$, so this decay mode by bubble nucleation is incompatible with the vacuum spin +structure, defined by phases of $e^{\pi R\sqrt{2}(\alpha/\sqrt{2}\mu)\gamma}$.

If we take $B^x = B^y = \alpha/\sqrt{2}\mu + 1/\sqrt{2}R$ and dimensionally reduce along $l'_x = l_x + 1/\sqrt{2}R$ and $l'_y = l_y + 1/\sqrt{2}R$, then the phase change about an orbit of $l'$ is found to be $-e^{\pi R(B\sqrt{2}-1/R)\gamma} = e^{\pi RB\sqrt{2}\gamma}$ which is consistent with the vacuum.

The action for this decay mode can be compared to (3.21); the transformation does not change the action, but after ensuring that the unit of charge is the same in each case, we find that

$$S_E(1,1) = 2S_E(1,0) \tag{3.26}$$

where the notation specifies the charges carried by the monopoles. Thence the rate of decay by creation of monopoles carrying $(1,1)$ charges is approximately half as large as the rate of decay by creation of monopoles carrying $(1,0)$ charges (of the same magnitude); as we would expect, the higher the charge, the smaller the rate of decay.

By applying a more general $O(6,22)$ transformation to these solutions, we might expect to obtain instantons describing the pair creation of other types of monopoles, such as H monopoles, within the backgrounds discussed in §2. Although a large class of solutions can be obtained by generating transformations, most of them will be singular and incomplete; the nature of the "dual" geometry depends on the fixed points of the isometry with respect to which we dualise and fixed points of the isometry in the original solution generically become singular points in the dual solution.

For example, the solution appropriate to H-monopole nucleation is given by the (Buscher) transformation (see [22] for a review of T-duality in string theory):

$$\begin{aligned} g_{55} &\rightarrow 1/g_{55} \\ A^1_{\bar{\psi}} &\rightarrow 0 \\ B_{5\bar{\psi}} &\rightarrow A^1_{\bar{\psi}} \end{aligned} \tag{3.27}$$

with all other fields invariant. Dualisation with respect to the isometry $\partial_x$ which has a fixed point set at $r = R$, $\theta = 0, \pi$ leads to a solution which is singular at these points (in both string and Einstein frames); thus we cannot interpret the solution as describing pair creation.

## 4 More general instantons

We have so far discussed only Ricci-flat instantons; evidently, more general decay modes of the vacuum involving non-zero gauge, antisymmetric tensor and dilaton fields should also be taken into account. Consistency with the background requires that all fields are asymptotically constant; these solutions were considered to some extent in [9] and we suggest generalisations here.

The decay modes presented in §3 involved five-dimensional Euclidean black hole solutions, with a non-trivial topology $R^2 \times S^3 \times T^5$ and in looking for generalisations it is natural to consider electrically charged black hole solutions in five dimensions. [1] In the following two sections, we work with the effective five-dimensional action, and implicitly take the product of the five-dimensional solution with a flat torus. Following the general prescription for dimensional reduction given in [23], we obtain from (2.1) an action in the string frame containing the terms:

$$S = \frac{1}{16\pi G_5} \int d^5x \sqrt{-G} e^{-\phi} \{R(G) + (\partial\phi)^2 - \frac{1}{12} H^2 - \frac{1}{4} F^2\} \tag{4.28}$$

with $F$ deriving from the left current algebra, and $G_5 = G_{10}/V$ where $V$ is the volume of the $T^5$. Rescaling to the Einstein frame $g_{ab} = e^{-2\phi/3} G_{ab}$, we obtain an action:

$$S = \frac{1}{16\pi G_5} \int d^5x \sqrt{g} \{R - \frac{1}{3}(\partial\phi)^2 - \frac{1}{12} e^{-\frac{4\phi}{3}} H^2 - \frac{1}{4} e^{-\frac{2\phi}{3}} F^2\} \tag{4.29}$$

We may now invoke Poincaré string-particle duality in five dimensions to relate the three form field strength to its dual:

$$e^{-\phi} H^{abc} = \frac{1}{2!\sqrt{-G}} \epsilon^{abcde} \bar{F}_{de} \tag{4.30}$$

which gives us the following action in terms of the axionic field strength $\bar{F}$:

$$S = \frac{1}{16\pi G_5} \int d^5x \sqrt{g} \{R - \frac{1}{3}(\partial\phi)^2 - \frac{1}{4} e^{\frac{4\phi}{3}} \bar{F}^2 - \frac{1}{4} e^{-\frac{2\phi}{3}} F^2\} \tag{4.31}$$

In §5 we shall consider more general solutions with both of these gauge fields non-trivial but we begin with a particularly simple five-dimensional (electrically) charged

[1] Five-dimensional black holes may carry a magnetic charge with respect to the three form field strength, but the latter takes the form $H = P\epsilon_3$, and does not asymptotically vanish, so we need not consider such solutions here.

black hole solution:

$$
\begin{aligned}
ds^2_{str} &= -\frac{(1-\frac{\mu}{r^2})}{(1+\frac{k\mu}{r^2})^2}dt^2 + \frac{dr^2}{(1-\frac{\mu}{r^2})} + r^2 d\Omega_3^2 \\
e^{-\phi} &= (1+\frac{k\mu}{r^2}) \\
A &= -\frac{1}{\sqrt{2}}\frac{\mu\mathrm{sinh}\delta}{(r^2+k\mu)}dt
\end{aligned}
\tag{4.32}
$$

where $k = (\cosh(\delta) - 1)/2$, and $A$ is the gauge potential associated with the field strength $F$. Such a solution was first constructed in [24], and the Euclidean section was discussed in [9]. We define the charge as:

$$
Q_f = \frac{1}{16\pi}\int *F \tag{4.33}
$$

and, with this convention, $Q_f = (\sqrt{2}/8)\pi\mu\mathrm{sinh}\delta$. We now look for a Euclidean section on which all the fields are real, by rotating $t \to i\tau$. To obtain a real gauge potential on the Euclidean section, we must also rotate $Q_f \to -iQ_f$ and hence $\mathrm{sinh}\delta \to -\mathrm{i}(\mathrm{sin}\delta)$, giving the solution:

$$
\begin{aligned}
ds^2_{str} &= \frac{(1-\frac{\mu}{r^2})}{(1-\frac{\kappa\mu}{r^2})^2}d\tau^2 + \{\frac{dr^2}{(1-\frac{\mu}{r^2})} + r^2 d\Omega_3^2\} \qquad (4.34) \\
A &= -\frac{1}{\sqrt{2}}\frac{\mathrm{sin}\delta(\mathrm{r}^2-\mu)}{(1-\kappa)(r^2-\kappa\mu)}d\tau
\end{aligned}
$$

where $\kappa = (1 - \mathrm{cos}\delta)/2$ and we have added a pure gauge term to the potential so that $A^2$ is non singular at $r = \sqrt{\mu}$. [2] The coordinate $r$ is now restricted to $r \geq \sqrt{\mu}$ and we must identify $\tau$ with period $\Delta\tau = 2\pi(1-\kappa)\sqrt{\mu}$. The limit $\kappa = 0$ corresponds to an uncharged solution, whilst in the limit $\kappa = 1$ the solution becomes singular. Since $0 \leq \kappa < 1$, the radius at infinity becomes smaller as $\kappa$ approaches its maximum value.

By rotating one of the coordinates on the sphere, we obtain:

$$
ds^2_{str} = \frac{(1-\frac{\mu}{r^2})}{(1-\frac{\kappa\mu}{r^2})^2}d\tau^2 + \frac{dr^2}{(1-\frac{\mu}{r^2})} + r^2\mathrm{cosh}^2\mathrm{t}(\mathrm{d}\Omega_2^2) - \mathrm{r}^2\mathrm{dt}^2 \tag{4.35}
$$

[2]Evidently the potential approaches a non-zero constant at infinity, and thence a (purely gauge) Maxwell potential must exist in the background also; this presents no problem since the most general supersymmetric vacua may have constant gauge potentials.

Since there are no terms of order $1/r$ in the fields, this solution has zero mass and charge, which is consistent with the fact that it results from the decay of a vacuum which certainly has zero mass and charge. Since the topology of the solution is $R^2 \times S^3$, and the Killing vector $\partial_\tau$ has a fixed point set of dimension three at $r = \sqrt{\mu}$, spinors must be antiperiodic about the imaginary time direction, which prevents the solution from describing the decay of a supersymmetric vacuum.

It is straightforward to calculate the Euclidean action for this solution; this is most easily done in the string frame, since we can convert the volume term (4.28) to a surface term [9] using the dilaton field equation:

$$S_E = -\frac{1}{8\pi G_5} \oint d^4x \sqrt{c} \{ (n \cdot \partial e^{-\phi}) + e^{-\phi} \mathcal{K} - e^{-\phi_0} \mathcal{K}_0 \} \tag{4.36}$$

where we now include the appropriate surface terms (and $\phi_0$ is the asymptotic value of the dilaton field). The action is thus:

$$S_E = \frac{\pi \mu}{8 G_4} (6\kappa + 1) \tag{4.37}$$

with $G_5 = G_4/\Delta\tau$; this is consistent with the Schwarzschild action given in §3 when $\kappa = 0$ as required. Re-expressing this in terms of the radius at infinity:

$$S_E = \frac{\pi R^2}{8 G_4} \frac{(6\kappa + 1)}{(1 - \kappa)^2} \tag{4.38}$$

As before, the decay rate $\Gamma \sim e^{-S_E}$ goes to zero as the supersymmetry breaking becomes arbitrarily small, and, for given radius $R$, the vacuum decay described by this solution is slower than decay via the Ricci-flat solutions of §3. If we let $\kappa \rightarrow 1$, with the radius $R$ finite, the action diverges, and the radius of the "horizon" approaches infinity. Letting $\kappa \rightarrow 1$ with $\mu$ constant gives a finite action, decreasing radius at infinity and an unstable horizon (since the horizon is singular for $\kappa = 1$). In the limit of small charges, $Q_f \ll \mu$, it is straightforward to show that:

$$S_E = \frac{\pi \mu}{8 G_4} (1 + \frac{48 Q_f^2}{\pi^2 \mu^2}) \tag{4.39}$$

with the radius at infinity approximately $\mu$.

We obtain the effective four-dimensional solution using the procedure given in [23] as:

$$G_{ab} = \begin{pmatrix} e^{\varphi} g_{\mu\nu} + G_{\tau\tau} A_\mu A_\nu & G_{\tau\tau} A_\nu \\ G_{\tau\tau} A_\nu & G_{\tau\tau} \end{pmatrix} \tag{4.40}$$

where $g_{\mu\nu}$ is the metric in the Einstein frame and $\varphi = \phi - \frac{1}{2}\mathrm{lnG}_{\tau\tau}$. Then the four-dimensional metric in the Einstein frame is:

$$\begin{aligned} ds^2_{ein} &= e^{-\varphi}\{\frac{dr^2}{(1-\frac{\mu}{r^2})} + r^2\mathrm{cosh}^2\mathrm{t}(\mathrm{d}\Omega_2^2) - \mathrm{r}^2\mathrm{dt}^2\} \\ e^{-2\varphi} &= (1-\frac{\mu}{r^2}) \end{aligned} \tag{4.41}$$

The solution describes the formation and subsequent expansion of a hole at $t = 0$, and differs from Witten's original decay mode [6] only by the presence of an additional scalar field in four dimensions (originating from $A_\tau$).

As in §3, we can also consider dimensional reduction along the Killing vector $\partial_\tau + B\partial_\phi$ where $B = n/R$. The four-dimensional fields obtained are:

$$\begin{aligned} ds^2_{ein} &= e^{-\varphi}\{\frac{dr^2}{(1-\frac{\mu}{r^2})} + r^2(d\theta^2 - \cos^2\theta\mathrm{dt}^2) + \mathrm{e}^{2\varphi}(\mathrm{r}^2-\mu)\sin^2\theta\mathrm{d}\psi^2\} \\ e^{-2\varphi} &= \{(1-\frac{\mu}{r^2}) + (1-\frac{\kappa\mu}{r^2})^2B^2r^2\sin^2\theta\} \\ A^1_\phi &= e^{2\varphi}(1-\frac{\kappa\mu}{r^2})^{-2}B^2r^2\sin^2\theta \\ \mathcal{A} &\equiv A_\tau = -\frac{\sin\delta}{\sqrt{2}(1-\kappa)}\frac{r^2-\mu}{r^2-\kappa\mu} \end{aligned} \tag{4.42}$$

describing the pair creation of monopoles within a background magnetic field $A^1$ and background scalar fields $\mathcal{A}$ and $\varphi$ (which are asymptotically constant). However, the magnetic field $B = n/R$ once again lies outside the range of validity $B \ll 1/R$, and we need to consider rotating black holes to obtain magnetic fields of physical magnitude.

For simplicity, we consider a five-dimensional black hole solution with only one electric charge $Q_f$, and one rotational parameter $a$ non-zero. Such a solution may be obtained from boosting the (Lorentzian) Myers-Perry solution; the most general such solutions are discussed in [25]. Rotating $t \to i\tau$, $a \to -i\alpha$ and $Q_f \to -iQ_f$, we obtain the Euclidean section (in the string frame):

$$ds^2_{str} = \Sigma\{\frac{(\Sigma-\mu)}{\Delta}d\tau^2 + \frac{dr^2}{(\rho^2-\alpha^2-\mu)} + d\theta^2 + \frac{\rho^2\cos^2\theta}{\Sigma}d\chi^2$$

$$
\begin{aligned}
& -\frac{\mu\alpha\sin^2\theta}{\Delta}(1+\cos\delta)\mathrm{d}\tau\mathrm{d}\psi+\frac{\sin^2\theta}{\Delta}(\Delta-\alpha^2\sin^2\theta(\Sigma+\mu\cos\delta))\mathrm{d}\psi^2\} \\
\Delta &= (\Sigma-\kappa\mu)^2 \\
e^{-2\phi} &= \frac{\Delta}{\Sigma^2} \qquad (4.43)\\
A_\tau &= -\frac{\sin\delta}{\sqrt{2}(1-\kappa)}\frac{\Sigma-\mu}{\Delta^{1/2}} \\
A_\psi &= -\frac{\mu\alpha\sin\delta\sin^2\theta}{\sqrt{2}\Delta^{1/2}}
\end{aligned}
$$

where $\Sigma = (\rho^2 - \alpha^2\cos^2\theta)$, $\kappa$ is defined as previously and we have included a pure gauge term in $A_\tau$ so that $A^2$ is non-singular at the poles of the horizon $\rho_h^2 = \mu + \alpha^2$. The charge $Q_f = (\sqrt{2}/8)\pi\mu\sin\delta$ using the same conventions as previously.

To avoid a conical singularity at the horizon, we choose the radius at infinity to be $R = (1-\kappa)/\rho_h$; the Euclidean angular velocity is $\Omega = \alpha/\mu(1-\kappa)$. The action is easily calculated from (4.36), with the background subtraction facilitated by the flatness of the $\mu = 0$ solution for all values of $\alpha$; then:

$$
S_E = \frac{\pi\mu(1+6\kappa)}{8G_4} = \frac{\pi R^2(1+6\kappa)}{8G_4(1-\kappa)^2(1-\Omega^2R^2)} \qquad (4.44)
$$

As usual, we rotate a coordinate on the sphere $\chi \to it$ to obtain the subsequent Lorentzian evolution. Then, dimensional reduction along $l = \partial_\tau + \Omega\partial_\psi$ leads to a bubble decay mode, and reduction along $l' = \partial_\tau + (\Omega + n/R)\partial_\psi$ describes monopole pair production, with the decay rate of the latter suppressed since the action is greater. For the latter, magnetic fields of physical magnitude and avoidance of conical singularities in the four-dimensional solution require $n = \pm 1$ and $|\Omega| \approx 1/R$.

We have considered only the most simple charged rotating solution; the prescription for obtaining the most general decay modes is as follows. Starting from the most general five-dimensional Lorentzian rotating, (electrically) charged black hole solution [25], we look for a Euclidean section on which all fields can be chosen to be real. If such a section exists, then by Witten rotating a coordinate on the sphere, we can obtain a vacuum decay mode. Dimensional reduction along a Killing vector with fixed point set of dimension three leads to decay by bubble formation, a decay process lying in a different superselection sector of the Hilbert space to the supersymmetric vacuum, whilst dimensional reduction along a Killing vector with a fixed point set of dimension one leads to decay by monopole pair production, a

decay process consistent with the spin structure of the background. Finally, by rotating the torus coordinates (allowing for non-trivial angles between the generating circles), we obtain the generalisations of the solutions discussed in §3.

All such decay modes do not describe the decay of the supersymmetric vacuum, are incomplete at null infinity, and have actions greater than the action of the original decay mode of Witten described in §3.

# 5 Extremal black holes as instantons

The discussion in the previous sections has been based around five-dimensional black hole solutions of topology $R^2 \times S^3$, whose asymptotic geometry is that of the background $R^4 \times S^1$. However, extremal black holes are believed to have the topology $S^1 \times R \times S^3$, with the Killing vector in the circle direction having no fixed point sets [26]. In contrast to the choice for non-extremal solutions we must choose a spin structure such that spinors are periodic in this direction, and there is hence no obstruction to the analytically continued solutions asymptotically admitting the covariantly constant spinors of the background.

To illustrate this, we consider a particular five-dimensional extremal black hole solution to the equations of motion which follow from (4.31), carrying electric charges with respect to both gauge fields where:

$$\begin{aligned} Q_{\bar{f}} &= \frac{1}{4\pi^2} \int * e^{\frac{4\phi}{3}} \bar{F} \\ Q_f &= \frac{1}{16\pi} \int * e^{-\frac{2\phi}{3}} F \end{aligned} \tag{5.45}$$

For a spherically symmetric solution we have:

$$\begin{aligned} * e^{\frac{4\phi}{3}} \bar{F} &= 2 Q_{\bar{f}} \epsilon_3 \\ * e^{-\frac{2\phi}{3}} F &= \frac{8 Q_f}{\pi} \epsilon_3 \end{aligned} \tag{5.46}$$

and there exist solutions with constant dilaton such that:

$$e^{2\phi} = 2\left(\frac{\pi Q_{\bar{f}}}{4 Q_f}\right)^2 \tag{5.47}$$

The field equations then imply that the metric takes the Reissner-Nordstrom form:

$$\begin{aligned} ds^2_{ein} &= -(1-(\frac{r_0}{r})^2)^2 dt^2 + (1-(\frac{r_0}{r})^2)^{-2} dr^2 + r^2 d\Omega_3^2 \\ r_0 &= (\frac{8Q_{\bar{f}}Q_f^2}{\pi^2})^{\frac{1}{6}} \end{aligned} \tag{5.48}$$

We consider these extremal solutions (with both charges non-zero) since extremal solutions with only one charge non-zero have degenerate horizons with zero area, and thus the Euclidean sections have naked singularities, and cannot be interpreted as an instantons. The above solution is the simplest extremal black hole with a non-degenerate horizon, and for this reason the corresponding dual solution in IIA theory compactified on $K3 \times T^2$ was recently discussed in the context of the microscopic description of the entropy [27].

We now attempt to analytically continue the Lorentzian solution into the Euclidean regime, by rotating $\tau = it$; now, the gauge fields in the original solution are:

$$\begin{aligned} \bar{F} &= \frac{16}{\pi^{4/3} r^3} Q_f^{4/3} Q_{\bar{f}}^{-1/3} dt \wedge dr \\ F &= \frac{8}{\pi^{1/3} r^3} Q_f^{1/3} Q_{\bar{f}}^{2/3} dt \wedge dr \end{aligned} \tag{5.49}$$

When we rotate $t \to i\tau$, if we impose the requirements that the dilaton field and $r_0^2$ are real, both $Q_f$ and $Q_{\bar{f}}$ remain real and positive, so that the gauge fields become pure imaginary. If however we impose the requirements that the gauge fields are real on the Euclidean section, then $r_0$ becomes complex, and the metric is not real. Thence a Euclidean section on which all the fields are purely real does not exist.

If we take (electric) field strengths that are pure imaginary on the Euclidean section, our solution takes the form:

$$\begin{aligned} ds^2_{ein} &= (1-(\frac{r_0}{r})^2)^2 d\tau^2 + (1-(\frac{r_0}{r})^2)^{-2} dr^2 + r^2 d\Omega_3^2 \\ \bar{F} &= i\frac{16}{\pi^{4/3} r^3} Q_f^{4/3} Q_{\bar{f}}^{-1/3} d\tau \wedge dr \\ F &= i\frac{8}{\pi^{1/3} r^3} Q_f^{1/3} Q_{\bar{f}}^{2/3} d\tau \wedge dr \end{aligned} \tag{5.50}$$

As before, we need to include (pure imaginary) gauge terms to the Maxwell potentials $A$ and $\bar{A}$ to ensure that both $A^2$ and $\bar{A}^2$ are non-singular at the horizon.

There is no naturally defined periodicity of $\tau$ and we define the periodicity at spatial infinity as $\beta$; the action [3] of the Euclidean solution (4.36) is then:

$$S_E = \frac{\pi r_0^2 \beta}{8G_5} = \frac{\pi r_0^2}{8G_4} \tag{5.51}$$

which implies a vanishing entropy in the semi-classical approach [28] since:

$$S = (\beta \partial_\beta - 1) S_E = 0 \tag{5.52}$$

(although string theory calculations give a non-zero answer and it is believed that string corrections as the length of the imaginary time direction approaches the string scale lead to a non-vanishing entropy as well as perhaps changing the topology to that of non-extremal solutions [29]).

For comparable values of the parameters $r_0$ and $\mu$, the decay rate by this mode is similar to that in §3; the range of validity of the semiclassical calculation requires that $r_0$ is much larger than the Planck length, and hence the rate of vacuum decay is necessarily slow. However, since $\beta$ is not fixed by the solution, we can choose a radius at infinity consistent with a Kaluza-Klein interpretation (unlike the solutions in the previous whose internal directions are too large for such interpretations).

As in [6], we construct a bubble decay mode of the vacuum by rotating a coordinate on the sphere. The mass on the initial value hypersurface vanishes, since the perturbation falls off faster than $1/r$, as do the charges of the fields; hence the solution may describe the decay of the vacuum state. The decay mode again involves the formation and subsequent expansion of a bubble of radius $r_0$, with null infinity incomplete.

Since there is no obvious obstruction to finding asymptotically constant spinors in this solution, it may seem at first sight as though the instanton represents a possible decay mode of the supersymmetric vacuum. However, the gauge fields remain imaginary in the Lorentzian continuation, and hence violate the dominant energy condition on the energy momentum tensor. Although the total charge is actually zero, there are non-vanishing pure imaginary Maxwell potentials. That is, although the solution has the requisite asymptotic spin structure for it to contribute to the decay of the vacuum, it is excluded by the unphysical behaviour of the energy momentum tensor.

[3] Since the metric is of the Reissner-Nordstrom form, the subtleties in calculating the boundary terms considered in [26] do not arise here.

More generally, *any* analytically continued extremal solution which asymptotically admits the constant spinors of the vacuum on a hypersurface of zero mass and charge must have an energy momentum tensor which does not satisfy the dominant energy condition. We justify this statement in the following section by considering the formulation of Witten's proof of the positive energy theorem in higher dimensions.

# 6 The positive energy theorem

In the examples that we have discussed so far, the instantons have fallen into three categories. Firstly, extremal black hole instantons admitting isometries with no fixed point sets have spin structures consistent with the background, but the energy momentum tensors of the analytically continued solutions do not satisfy the dominant energy condition (at least in the example we gave). Secondly, in non-extremal black hole instantons, if we consider dimensional reduction along a Killing vector admitting an eight dimensional fixed point set, spinors must be antiperiodic about an orbit at infinity of this isometry, and hence the solution lies in a different superselection sector of the Hilbert space to the vacuum. Thirdly, again in non-extremal black hole instantons, if we consider dimensional reduction along a Killing vector admitting a six-dimensional fixed point set, we obtain a solution consistent with the decay of a magnetic vacuum in four dimensions. We now discuss more generally the existence of instantons, and show that there are no decay modes consistent both with the dominant energy condition and with the supersymmetric spin structure of the background.

Yang-Mills instantons do not indicate a possible decay mode of the vacuum since there does not exist a surface of constant time from which we can continue them as real Yang-Mills fields in Minkowski space. The analogue for ten-dimensional supergravity would be a nine-dimensional surface with zero second fundamental form; such a surface acts as a "turning" point at which the instanton matches the space into which the vacuum decays.

In the previous sections, starting from a Lorentzian spacetime we constructed Euclidean solutions by rotating $(t, x^m) \rightarrow (i\tau, x^m)$. Even if the original Lorentzian solution did not admit a hypersurface of constant time of zero momentum we could find a Euclidean section on which the metric was real, but only by analytically continuing a momentum parameter in the solution. When we look for a Lorentzian

section of a Euclidean solution, to describe the subsequent evolution from an initial value hypersurface, we cannot analytically continue momentum parameters and such a section will only exist if there is a zero momentum hypersurface.

Starting from a ten-dimensional Euclidean solution $g_{MN}$ which admits a nine-dimensional complete hypersurface $\Sigma$ with induced metric $\hat{g}_{MN}$, we analytically continue back to a Lorentzian signature by rotating $(\tau, x^m) \to (-it, x^m)$. We choose the 0-vector to be orthogonal to $\Sigma$ and, on the Lorentzian section, we have a unit timelike normal to $\Sigma$ given by $t_M = g_{0M}/\sqrt{-g_{00}}$ with the induced metric being:

$$\hat{g}_{MN} = g_{MN} - t_M t_N \tag{6.53}$$

and the second fundamental form of $\Sigma$ is:

$$\mathcal{K}_{MN} = t_{(P;Q)} \hat{g}^Q_M \hat{g}^P_N \tag{6.54}$$

Reality of the metric on the Lorentzian section requires that the $g_{0m}$ terms vanish, which implies that $\hat{g}_{0m} = \mathcal{K}_{00} = \mathcal{K}_{0m} = 0$ and:

$$\mathcal{K}_{mn} = \frac{1}{2\sqrt{-g_{00}}} g_{nm,0} \tag{6.55}$$

The surface $\tau = \tau_0$ of the Euclidean solution must match that the surface $t = t_0 = i\tau_0$ of the analytically continued solution; a necessary condition is that $g_{nm,0}|_{t_0} = 0$. Hence, if an instanton is to describe the decay of the vacuum, it must admit a surface with zero extrinsic curvature from which we can analytically continue the solution. We usually find such a surface by looking for a hypersurface orthogonal Killing vector in the Euclidean solution and Wick rotating.

Now, if one has such a hypersurface of constant time, Witten's proof of the positive energy theorem can be applied unless there is some obstruction such as a black hole. Suppose that we have an instanton for which the fixed point sets of the isometries do not prevent us from finding asymptotically constant spinors; the solution then lies in the same superselection sector as the vacuum. However, the absence of any obstruction on the initial value hypersurface allows us to prove the positivity of the mass by Witten's method; that is, if the solution is not flat, the mass must be positive, and the vacuum cannot decay into this state.

Putting it this way makes it sound as though the decay modes of the supersymmetric vacuum must be trivial; if a black hole type of obstruction is present, the positive energy theorem does not apply but the instanton does not lie in the same

superselection sector of fermions as the vacuum. In the absence of an obstruction the positive energy theorem applies and prevents the existence of instantonic decay modes unless the Witten proof fails in another (unphysical) way such as the dominant energy condition breaking down, which we discussed in §5. We now discuss the formulation of the positive energy theorem for spacetimes asymptotic to $M^4 \times T^6$.

We consider an asymptotically flat solution to the Euclidean field equations derived from the Einstein frame action in ten dimensions (2.2); the graviton field equation gives:

$$R_{MN} - \frac{1}{2} R g_{MN} = 8\pi G_{10} T_{MN} \tag{6.56}$$

where the energy momentum tensor includes contributions from the dilaton, antisymmetric tensor and gauge fields. The field configuration must also be consistent with the other constraint equations derived from the action.

We then analytically continue the solution to obtain a Lorentzian solution, imposing the condition that the energy momentum tensor satisfies the dominant energy condition [30]; that is, the local energy density $T_{00}$ is positive (or zero) at each point in the Lorentzian spacetime and in each local Lorentz frame. The total energy momentum tensor is:

$$\begin{aligned} T_{MN} &= \frac{1}{8}\{(\partial_M \Phi)(\partial_N \Phi) - \frac{1}{2}(\partial\Phi)^2 g_{MN}\} + \frac{1}{4} e^{-\frac{\Phi}{2}}\{g_{NN'} H_{MPQ} H^{N'PQ} - \frac{1}{6} H^2 g_{MN}\} \\ &\quad + \frac{1}{2} e^{-\frac{\Phi}{4}}\{g_{NN'} F^{(f)}_{MP} F^{NP}_{(f)} - \frac{1}{4} Tr(F^2) g_{MN}\} \end{aligned} \tag{6.57}$$

It is well known that each of the contributions to the energy momentum tensor obey the required condition, provided that the fields are real. Since this condition is critical to the proof, we include a short discussion of the dominant energy condition in the appendix.

Evidently reality of the fields is a non-trivial constraint when we consider analytic continuation of Euclidean solutions. F will be real in Lorentzian spacetime provided that $F_{\tau m}$ is pure imaginary and $F_{mn}$ is pure real; that is, the magnetic field must be real and the electric field imaginary on the Euclidean section. Similarly, the reality of H in the Lorentzian spacetime is ensured by $H_{\tau mn}$ being pure imaginary, and $H_{mnp}$ being pure real.

We choose to work in the Einstein frame since the energy momentum tensor in the string frame does not satisfy the dominant energy condition [24], [31]; the

graviton field equation obtained from the string frame action (2.1) is:

$$R_{MN}(G) - \frac{1}{2}R(G)G_{MN} = 8\pi G_{10}T^{str}_{MN} \tag{6.58}$$

where the total energy momentum tensor is defined as:

$$\begin{aligned} T^{str}_{MN} &= -2\{(\partial_M\Phi)(\partial_N\Phi) - \frac{1}{2}(\partial\Phi)^2 G_{MN}\} + \frac{1}{4}\{G_{NN'}H_{MPQ}H^{N'PQ} - \frac{1}{6}H^2 G_{MN}\} \\ &\quad + \frac{1}{2}\{G_{NN'}F^{(f)}_{MP}F(f)^{N'P} - \frac{1}{4}Tr(F^2)G_{MN}\} \end{aligned} \tag{6.59}$$

Hence, the dilaton contribution to the energy momentum tensor no longer satisfies the dominant energy condition, because of the change of sign of the $(\partial\Phi)^2$ term in the action.

Since the solution is asymptotically flat, we can decompose the metric at spatial infinity as:

$$g_{\hat{M}\hat{N}} = \begin{pmatrix} -\delta_{\hat{0}\hat{0}} + h_{\hat{0}\hat{0}}(x^p) & h_{\hat{0}\hat{m}}(x^p) \\ h_{\hat{n}\hat{0}}(x^p) & \delta_{\hat{m}\hat{n}} + h_{\hat{m}\hat{n}}(x^p) \end{pmatrix} \tag{6.60}$$

where the $h_{\hat{0}\hat{m}}$ terms vanish if $\Sigma$ has zero momentum. We use asymptotically flat coordinates, and work in an orthonormal frame. If the perturbation to the flat metric is of order $(1/r^k)$; derivative terms are of order $1/r^k$ for $\partial_i$ terms and of order $1/r^{(k+1)}$ for $\partial_\mu$ terms.

For a solution admitting a hypersurface of asymptotic geometry $R^3 \times T^6$ the ADM energy can be expressed as [32]:

$$E_{ADM} = \frac{1}{16\pi G_{10}}\oint_\infty d\Sigma^m(\partial_n h_{nm} - \partial_m h_{nn}) \tag{6.61}$$

where the integral is taken over a boundary at infinity of $\Sigma$; the ADM momentum is given by:

$$P_{\hat{m}} = \frac{1}{16\pi G_{10}}\oint_\infty d\Sigma^{\hat{n}}(\partial_{\hat{n}}h_{\hat{0}\hat{m}} - \partial_{\hat{0}}h_{\hat{n}\hat{m}} + \delta_{\hat{n}\hat{m}}\partial_{\hat{0}}h_{\hat{p}\hat{p}} - \delta_{\hat{n}\hat{m}}\partial_{\hat{p}}h_{\hat{0}\hat{p}}) \tag{6.62}$$

Expanding the expression for the ADM energy in terms of the torus coordinates $x^i$ and the external coordinates $x^\alpha$:

$$E_{ADM} = \frac{1}{16\pi G_{10}}\oint_\infty d\Sigma^\alpha\{\partial_\beta h_{\beta\alpha} - \partial_\alpha h_{ii} - \partial_\alpha h_{\beta\beta}\} \tag{6.63}$$

where the $\partial_i h_{i\alpha}$ terms vanish when we consider a solution admitting a global $U(1)^6$ isometry group. Since the volume element is of order $r^2$, the energy is only well-defined when the integrand is of order $1/r^2$ which requires that $h_{\alpha\beta}, h_{ii}$ are of order $1/r$. Since the functions $h_{i\alpha}$ must be periodic in the coordinates $x^i$, even if they are not independent of the torus coordinates, the expression for the ADM energy in (6.63) is valid even when the metric perturbations are dependent on the torus coordinates.

To prove the positive mass theorem for solutions which tend to the required background we consider spinors obeying a Dirac type equation on the nine-dimensional hypersurface. Our discussion follows closely that of [33], and although we are interested in zero momentum hypersurfaces for generality we do not impose the requirement that $\mathcal{K} = 0$, since it is not required by the proof. Projecting the ten-dimensional covariant derivative $D_M$ onto the hypersurface:

$$D_{\hat{m}}\epsilon = (\nabla_{\hat{m}} + \frac{1}{2}\mathcal{K}_{\hat{m}\hat{n}}\gamma^{\hat{n}}\gamma^{\hat{0}})\epsilon \tag{6.64}$$

where $\nabla$ is the covariant derivative on the hypersurface, the gamma matrices are constructed from the 32-dimensional spinor representation of $SO(9,1)$ and $D_M = \partial_M + \Gamma_M$ with $\Gamma_M$ the spin connection matrices. Then, multiplying by $\gamma^{\hat{m}}$, we obtain the Witten equation:

$$\gamma^{\hat{m}}\nabla_{\hat{m}}\epsilon = -\frac{1}{2}\mathcal{K}\gamma^{\hat{0}}\epsilon \tag{6.65}$$

with $\mathcal{K}$ the trace of the second fundamental form. If we multiply by $\epsilon^*$, act on the result with $\gamma^{\hat{m}}\nabla_{\hat{m}}$ and use the Ricci identity, we obtain:

$$\begin{aligned}\nabla_m(\epsilon^* D^m\epsilon) \;&=\; (D_m\epsilon)^*(D^m\epsilon) + \frac{1}{4}\epsilon^*\{\mathcal{R} + \mathcal{K}^2 - \mathcal{K}_{mn}\mathcal{K}^{mn} \\ &\quad +2\nabla_m(\mathcal{K}^{mn} - \hat{g}^{mn}\mathcal{K})\gamma_n\gamma^{\hat{0}}\}\epsilon\end{aligned} \tag{6.66}$$

where $\mathcal{R}$ is the Ricci scalar on $\Sigma$ of the induced metric. The field equations are:

$$\begin{aligned}\mathcal{R} + \mathcal{K}^2 - \mathcal{K}_{mn}\mathcal{K}^{mn} \;&=\; 16\pi G_{10}T_{\hat{0}\hat{0}} \\ \nabla_m(\mathcal{K}^{mn} - h^{mn}\mathcal{K}) \;&=\; 8\pi G_{10}T^{\hat{0}n}\end{aligned} \tag{6.67}$$

and since the total stress energy tensor satisfies the dominant energy condition implying that $T_{\hat{0}\hat{0}} \geq |T_{\hat{m}\hat{n}}|$:

$$\nabla_m(\epsilon^* D^m\epsilon) \geq (D_m\epsilon)^*(D^m\epsilon) \tag{6.68}$$

Upon integrating this over the initial value hypersurface, we obtain:

$$\oint_{\infty} \epsilon^* D_m \epsilon d\Sigma^m - \oint_{H} \epsilon^* D_m \epsilon d\Sigma^m \geq \int_{\Sigma} (D_m \epsilon)^* (D^m \epsilon) d\Sigma \tag{6.69}$$

where we integrate over the region of $\Sigma$ bounded by an inner surface $H$ and a surface at infinity. In the following, we shall assume that the inner surface term vanishes; this is certainly true if $H$ is an apparent horizon or a minimal surface in a maximal hypersurface (since the proofs given in [33] are easily generalised to ten dimensions). This covers all cases of inner boundaries in which we are interested.

Let the surface term at infinity be $\mathcal{S}$; since the right-hand side is positive semi-definite, $\mathcal{S}$ is also positive semi-definite and is an invariant of the initial value hypersurface. The contributing terms to the integrand are of order $1/r^2$, with the $1/r$ contributions vanishing as:

$$\oint_{\infty} dx^i (\partial_i h_{mn}) = \oint_{\infty} dx^i \partial_i \epsilon = 0 \tag{6.70}$$

even when $h_{mn}$ is $x^i$ dependent, since any such dependence must be periodic in $x^i$. In fact, if we assume that there are no Kaluza-Klein type charges arising from the torus, then $\partial_i h_{mn}$ and $h_{\mu i}$ fall off as $1/r^2$ and the leading order deviation will be independent of these terms in any case.

We now demonstrate the relationship between $\mathcal{S}$ and the energy by considering solutions to Witten's equation; we omit much of the analysis since it follows as a direct generalisation of that in [8]. Now, no non-zero spinor satisfying Witten's equation on the initial value hypersurface vanishes as $r \to \infty$. This follows directly from (6.69) since if we assume that a solution to Witten's equation which vanishes at infinity exists, the left hand side of (6.69) vanishes, as such a solution must decay at least as fast as $1/r^2$ (for asymptotic geometry $R^3 \times T^6$). The energy-momentum tensor then vanishes, and $D_m \epsilon \equiv 0$; if however $D_m \epsilon \equiv 0$ and $\epsilon \neq 0$, then $\epsilon$ does not vanish at infinity, thus completing the proof.

We consider a solution to the Witten equation asymptotically approaching a constant spinor $\epsilon_0$; the asymptotic geometry constrains $\epsilon - \epsilon_0 = \tilde{\epsilon}(\theta, \phi, x^i)/r)$. The existence of such solutions is non-trivial even in the four-dimensional case, and was the subject of [34] and [35]. Here, however, such spinors exist by assumption, since we are only interested in solutions lying in the same superselection sector of the Hilbert space as the vacuum, requiring that the constant spinors of the vacuum are asymptotically admitted. Since $\epsilon$ is a solution of the Witten equation (6.65), we find that:

$$\tilde{\epsilon} = lim_{r \to \infty} \{ r^2 \gamma^r \gamma^m \Gamma_m \epsilon_0 + r \gamma^r (\gamma^\alpha \partial_\alpha) \tilde{\epsilon} \} \tag{6.71}$$

where we use only the $1/r^2$ terms in $\Gamma_m$. Then substituting $\epsilon$ in the expression for $\mathcal{S}$:

$$\mathcal{S} = \oint_{\infty} d\Sigma^{\alpha}\{\epsilon_0^* \Gamma_{\alpha}\epsilon_0 + \epsilon_0^* \partial_{\alpha}(\frac{\tilde{\epsilon}}{r})\} \tag{6.72}$$

where we again retain only terms of order $1/r^2$ in the integrand. Using (6.71), we find that:

$$\oint_{\infty} d\Sigma^r \epsilon_0^* \frac{\tilde{\epsilon}}{r^2} = \oint_{\infty} d\Sigma^r \{\epsilon_0^* \gamma^r \gamma^m \Gamma_m \epsilon_0 + \epsilon_0^* \frac{\gamma^r}{r^3}(\gamma^{\alpha}\partial_{\alpha})\tilde{\epsilon}\} \tag{6.73}$$

In the limit of spatial infinity only the first term contributes to (6.72)and hence:

$$\mathcal{S} = \oint_{\infty} d\Sigma^{\hat{\alpha}} \epsilon_0^* (\Gamma_{\hat{\alpha}} - \gamma_{\hat{\alpha}}\gamma^{\hat{n}}\Gamma_{\hat{n}})\epsilon_0 \tag{6.74}$$

where we retain only terms of order $1/r^2$ in $\Gamma_{\hat{m}}$. Now the linearised spin connection matrices are defined by:

$$\Gamma_m = \frac{1}{16}\{\partial_N h_{Mm} - \partial_M h_{Nm}\}[\gamma^M, \gamma^N] \tag{6.75}$$

and hence substituting for $\Gamma$ and comparing with the expressions for the ADM energy and momentum, we obtain the relationship:

$$\mathcal{S} = 4\pi G_{10}\epsilon_0^*(E_{ADM} + P_{\hat{m}}\gamma^{\hat{m}}\gamma^{\hat{0}})\epsilon_0 \tag{6.76}$$

The non-negativity of $\mathcal{S}$ then implies that $E_{ADM} \geq \left|P^{\hat{m}}\right|$; thence the mass is positive semi-definite, vanishing if and only if $D_m\epsilon = 0$ and the energy momentum tensor is null. The integrability condition on the spinors and the existence of a basis of such covariantly constant spinors then imply that $R_{mnMN} = 0$, with Einstein's equations leading to $T_{MN} = 0$ and thus $R_{MNPQ} = 0$. That is, the mass vanishes if and only if the spacetime is flat, and there is no state with the requisite spin structure into which the vacuum can decay. As usual, by replacing the covariant derivative with a modified derivative dependent on the gauge fields, Bogomolnyi bounds relating the mass and charges may be derived.

We next discuss the ways in which the solutions given in the previous sections "evade" the positive energy theorem. Suppose that we consider a metric on the hypersurface with a leading order perturbation of $1/r^k$ from the flat metric with $k > 1$; then the energy (and momentum) defined in (6.63) necessarily vanishes. If massless

Witten spinors are admitted on the hypersurface, and the energy momentum tensor satisfies the dominant energy condition, (6.66) must be satisfied and thus:

$$\int_{\Sigma} \{(D^m \epsilon^*)(D_m \epsilon) + 4\pi G_{10} \epsilon^* (T_{\hat{0}\hat{0}} + T^{\hat{0}m} \gamma_m \gamma^{\hat{0}}) \epsilon\} d\Sigma = 0 \tag{6.77}$$

This integral over the hypersurface will vanish if and only if $D_m \epsilon \equiv 0$ and the energy momentum tensor is null, which implies that the spacetime is flat. Hence, if we have a metric on the hypersurface which approaches the background faster than $1/r$, and the dominant energy condition is satisfied by the (non-flat) solution, we will not be able to find asymptotically constant solutions to Witten's equation; the topology change implicit in these perturbations implies an obstruction preventing the existence of such spinors.

The five-dimensional black hole solutions considered in §3 and §4 evade the positive energy theorem in this way; the masses of the analytically continued solutions vanish, but there exist no asymptotically constant spinors on the initial value hypersurfaces.

We are now able to justify the statement in §5 that the positive energy theorem implies that there exists no Euclidean section of certain extreme black hole solutions on which all fields are real. Suppose we have an analytically continued solution admitting a hypersurface whose mass and charges vanish, as the fields decay at infinity as $1/r^2$. The horizon in the original solution must be non-singular, so that this hypersurface is non-singular. There is no topological obstruction to finding solutions to the Witten equation on the initial value hypersurface, and we can show that:

$$\oint_{\infty} (\epsilon^* D_m \epsilon) d\Sigma^m - \oint_{H} (\epsilon^* D_m \epsilon) d\Sigma^m = 0 \tag{6.78}$$

where we must take account of the inner boundary $H$. Following [33], we choose $\epsilon$ to satsify the constraint equation $\gamma^{\hat{0}} \gamma^{\hat{1}} \epsilon = \epsilon$ (with the one direction normal to $H$) and thus restrict the frredom of $\epsilon$ on $H$ by half, as required. We may then show that the only contributing inner boundary term is:

$$-\oint_{H} (\epsilon^* D_m \epsilon) d\Sigma^m = -2 \oint_{H} (\epsilon^* \epsilon) \mathcal{J} d\Sigma_h \tag{6.79}$$

where $\mathcal{J}$ is the trace of the second fundamental form of $H$ embedded in the hypersurface $\Sigma$. Now for any such extremal black hole solution the second fundamental form of $H$ in the hypersurface vanishes, and hence the inner boundary term must vanish.

However, equation (6.78) contradicts (6.69), whose right hand side is positive definite for a non-flat solution. Since the derivation assumed that the energy momentum tensor satisfies the dominant energy condition we conclude that this condition must break down, and hence the gauge field strengths must be imaginary in this Lorentzian continuation. That is, although the solution has the requisite asymptotic spin structure for it to contribute to the decay of the vacuum, it is excluded by the non physical behaviour of the energy momentum tensor. Hence, if an extremal black hole solution with non-degenerate horizon admits a Euclidean section whose topology is consistent with asymptotically constant spinors, it is impossible to find an analytic continuation on which all fields are real.

Another class of solutions which evade the positive energy theorem are those obtained from taking the product of Euclidean black p-branes of topology $R^{2+p} \times S^2$ with a flat time direction (and flat circle directions) [14]. The effective four-dimensional solutions describe either the a static bubble or a static monopole pair within background fields and are excluded from the proofs of the positive energy theorem by spin structure arguments. It is however easy to show that the resultant configurations are unstable (because of the negative modes of the Euclidean p-brane solutions)and in any case such decay modes are excluded by their spin structures.

Therefore, to contribute to the decay of the vacuum, a Euclidean instanton must have a section on which the metric may be analytically continued to the Lorentzian regime. A necessary and sufficient condition is the existence of a hypersurface of zero second fundamental form. If the hypersurface does not admit the same number of asymptotically constant spinors as the vacuum, it will not describe the decay of the supersymmetric vacuum. If the hypersurface does admit asymptotically constant spinors, they must be covariantly constant in order that the mass is zero. If we have the required number of constant spinors, the hypersurface must be precisely $R^3 \times T^6$.

There are hence no instantons contributing to the decay rate of the supersymmetric vacuum. *Bubble* decay modes are inconsistent with the background spin structure; *monopole* decay modes describe the decay of an (unphysical) magnetic vacuum and *extremal black hole* instantons are inconsistent with the dominant energy condition.

# 7 Calabi-Yau compactification

The discussions in the previous sections applied to toroidal compactifications of the heterotic string theory; compactification on a Calabi-Yau space gives rise to a theory with (more realistically) $N = 1$ supersymmetry in four dimensions. The spectrum of the theory is certainly stable since there are no modes with imaginary frequencies [1] and the semi-classical stability of the vacuum will be determined by whether instanton solutions to the classical field equations exist.

We may immediately exclude the possibility that the Calabi-Yau vacuum can decay into another supersymmetric state; the supersymmetry will require that on an initial value hypersurface two covariantly constant spinors are admitted on the compact space and four are admitted on the external space. If however such spinors are admitted globally throughout the hypersurface, the hypersurface must be precisely $R^3 \times K_{SU(3)}$, and the spacetime must be the vacuum. One might imagine that there exist instantons corresponding to the tunnelling between different $K_i$; however, cobordism theory requires that the $K_i$ must have the same characteristic Stiefel-Whitney and Chern numbers [10] and thence are topologically indistinct. Thus vacuum instability can again result only from the consideration of non-supersymmetric states which are not simple metric products but rather contain topological defects.

An instanton will only describe a possible decay mode of the vacuum if it admits a nine-dimensional hypersurface of zero second fundamental form whose geometry asymptotically approaches that of $R^3 \times K_{SU(3)}$. At infinity one would want there to be two asymptotically covariantly constant spinors on the Calabi-Yau and four on the $R^3$ as for the vacuum. The most general decay modes may have non-vanishing antisymmetric tensor and dilaton fields which are asymptotically constant; the energy momentum tensor again satisfies the dominant energy condition provided that the fields are real on the Lorentzian section.

Now, the generalised expression for the ADM energy of a solution $g_{MN}$ with respect to a background solution ${}^{\circ}g_{MN}$ is [32]:

$$E_{ADM} = \frac{1}{16\pi G_{10}} \oint_{\infty} d\Sigma^m \{ {}^{\circ}D_n g_{mp} - {}^{\circ}D_m g_{np} \} {}^{\circ}g^{np} \tag{7.80}$$

where ${}^{\circ}D_m$ is the covariant derivative in the background. We assume that $g_{MN}$ is an analytically continued solution to (6.56), with the energy momentum tensor satisfying the dominant energy condition, and the field configuration consistent with

the other constraint equations. We decompose the metric at spatial infinity as:

$$g_{MN} = \begin{pmatrix} {}^\circ g_{00} + h_{00}(x^p) & h_{0m}(x^p) \\ h_{n0}(x^p) & {}^\circ g_{mn} + h_{mn}(x^p) \end{pmatrix} \tag{7.81}$$

with $h_{MN}$ decaying as $1/r$ and $h_{0m}$ terms vanishing for a zero momentum hypersurface. Then we can rewrite the ADM energy as:

$$E_{ADM} = \frac{1}{16\pi G_{10}} \oint_\infty d\Sigma^\alpha \{\partial_\beta h_{\alpha\beta} - \partial_\alpha h_{\beta\beta} + {}^\circ g^{ij}({}^\circ D_i h_{\alpha j}) - {}^\circ g^{ij}\partial_\alpha h_{ij}\} \tag{7.82}$$

We consider solutions to the Witten equation (6.65) on the hypersurface now approaching *covariantly* constant spinors; that is we look for a solution $\epsilon$ approaching $\epsilon_0$ where:

$$^\circ D_m \epsilon_0 = 0 \tag{7.83}$$

and $\epsilon$ approaches $\epsilon_0$ as $1/r^2$. For such a solution, it follows from the dominant energy condition that:

$$\oint_\infty \epsilon^* D_m \epsilon \geq \int_\Sigma (D_m \epsilon)^*(D^m \epsilon) \tag{7.84}$$

where we assume the only boundary is at infinity. Since $\epsilon$ satisfies the Witten equation, it is straightforward to show that the only contributing terms to the invariant $\mathcal{S}$ give:

$$\mathcal{S} = \oint_\infty d\Sigma^\alpha \epsilon_0^*(\Gamma_\alpha - \gamma_\alpha \gamma^m \Gamma_m)\epsilon_0 \tag{7.85}$$

where we linearise the covariant derivative about the background as $D_m = {}^\circ D_m + \Gamma_m$, and:

$$\Gamma_m = \frac{1}{16}({}^\circ D_N h_{mM} - {}^\circ D_m H_{MN})[\gamma^M, \gamma^N] \tag{7.86}$$

Substituting into (7.85), and following the same steps as in [8], we find that:

$$\begin{aligned} \mathcal{S} &= \oint_\infty d\Sigma^\alpha \epsilon_0^* \epsilon_0 \{({}^\circ D_m h_{\alpha n}){}^\circ g^{mn} - ({}^\circ D_\alpha h_{mn}){}^\circ g^{mn}\} \\ &= 16\pi G_{10} \epsilon_0^* \epsilon_0 E_{ADM} \end{aligned} \tag{7.87}$$

where we assume that the hypersurface has zero second fundamental form. Since $\mathcal{S}$ is positive semi-definite, the ADM energy of any solution asymptotically approaching this background is also constrained to be positive semi-definite with respect to the background. Vanishing of the energy requires that the energy momentum tensor vanishes, and $D_m \epsilon_0 = 0$. Since the background admits a basis of covariantly constant

spinors, and $\epsilon_0$ is an arbitrary element of the basis, zero energy requires the existence of the full number of covariantly constant spinors on the hypersurface, forcing the hypersurface to be precisely $R^3 \times K_{SU(3)}$.

Even if we assume that the perturbations fall away sufficiently quickly that the energy vanishes, then the vanishing of $\mathcal{S}$ implies that the requisite solutions to Witten's equation (6.65) are only admitted if $D_m \epsilon \equiv 0$, again implying that the solution is precisely the background. Even if other solutions of zero energy with respect to the background exist, they cannot have the required asymptotic spin structure. Obstructions on the hypersurface may allow the evasion of the positive energy theorem, but also imply either an incompatible spin structure at infinity or a violation of the dominant energy condition. Since the instanton cannot contribute to the decay rate unless one has asymptotically the requisite spinors, we conclude that there can be no instantonic decay modes of the Calabi-Yau vacuum. It is similarly straightforward to prove the absence of decay modes of the $K3 \times T^2$ vacuum using the basis of 16 covariantly constant spinors.

# 8 Eleven-dimensional supergravity

Our discussions so far have been restricted to ten dimensional supergravity but the same arguments can be applied to eleven-dimensional supergravity. The bosonic sector of the action of $N = 1$ supergravity in eleven dimensions is given by [36]:

$$S = \int d^{11} x \mathcal{E} \{ \frac{1}{4} R(\mathcal{E}) - \frac{1}{48} \mathcal{F}^2 + \frac{2}{(12)^4} \epsilon^{x_1 \dots x_{11}} \mathcal{F}_{x_1 \dots x_4} \mathcal{F}_{x_5 \dots x_8} \mathcal{A}_{x_9 \dots x_{11}} \} \tag{8.88}$$

where $\mathcal{E}$ is the elfbein, and $\mathcal{F}_{wxyz} = 4\partial_{[w} \mathcal{A}_{xyz]}$. The conjectured duality between eleven dimensional supergravity compactified on a circle and strongly coupled type IIA superstring theory [37] (as well as other related dualities) suggests that the absence of instantonic decay modes of the former should for consistency imply the absence of such decays for the latter.

The equations of motion derived from (8.88) (assuming that the gravitino vanishes) are:

$$\begin{aligned} R_{yz} - \frac{1}{2} R g_{yz} &= \frac{1}{3} \{ \mathcal{F}_{y x_1 x_2 x_3} \mathcal{F}_z^{\ x_1 x_2 x_3} - \frac{1}{8} g_{yz} \mathcal{F}^2 \} \\ D_y \mathcal{F}^{y x_1 x_2 x_3} &= -\frac{1}{576} \epsilon^{x_1 \dots x_{11}} \mathcal{F}_{x_4 \dots x_7} \mathcal{F}_{x_8 \dots x_{11}} \end{aligned} \tag{8.89}$$

Now, if we require a vacuum state in which the three form vanishes, the solution must admit at least one covariantly constant spinor in order to preserve some supersymmetry. In the context of compactification to four dimensions, this implies that the holonomy of the compact space must be a subgroup of $Spin(7)$ [2]. We usually take the compact space to be $T^7$, $K3 \times T^3$, $K_{SU(3)} \times S^1$ or $K_{Spin(7)}$ for which the eleven dimensional vacuum admits 32, 16, 8 or 4 covariantly constant spinors respectively.

We consider the existence of instantons by, as usual, looking for solutions to the Euclidean equations of motion whose asymptotic geometry is that of the background. The most general such solution may have a non-zero three form field which is asymptotically constant, and consistent with the field equations.

Such a solution only contributes to the vacuum decay rate if it both admits an initial value hypersurface from which we can describe the subsequent Lorentzian evolution and also asymptotically admits the requisite number of covariantly constant spinors of the vacuum on this hypersurface. In addition, we require that the analytic continuation of the four form field strength $\mathcal{F}$ is real; that is, on the Euclidean section, the "electric" part of $\mathcal{F}$ is imaginary and the "magnetic" part is real. Then, the energy momentum tensor satisfies the dominant energy condition (see the appendix) and the spinorial proofs of the positive energy theorem may be applied.

However, if all of these requirements are satisfied, the methods of §6 can be used to show that the energy of the solution with respect to the background is positive semi-definite and only vanishes if the solution is identical to the background. That is, there are no states into which the vacuum can decay, consistent with asymptotically admitting the covariantly constant spinors of the vacuum. For compactifications admitting at least one circle factor, we will of course be able to find bubble and monopole decay modes of a non-supersymmetric vacuum.

# 9 Conclusions

We conclude by recapitulating the arguments by which we exclude decay modes of a supersymmetric vacuum solution of supergravity theories in ten and eleven dimensions. To contribute to the decay of the vacuum, a Euclidean instanton must have a section on which the metric can be analytically continued to the Lorentzian regime. A necessary and sufficient condition is the existence of a hypersurface of zero second fundamental form. If this hypersurface does not admit the same number of

asymptotically constant spinors of the vacuum, it will not describe the decay of the supersymmetric vacuum. If the hypersurface does admit asymptotically constant spinors, they must be covariantly constant in order that the mass is zero (unless we allow the energy momentum tensor to be unphysical and violate the dominant energy condition). If we have the required number of covariantly constant spinors, the hypersurface must be precisely the vacuum state. Thence, there is no state into which the vacuum can decay.

All instantons "evade" the positive energy theorem either by violating the dominant energy condition, or by having an incompatible spin structure, or by describing the decay of a four-dimensional magnetic vacuum. The latter decay modes are consistent with fermions, but involve unphysical fields, and unconventional identifications on the internal torus. In addition the sizes of the internal directions are fixed by the solutions and are necessarily too large for a clear Kaluza-Klein interpretation.

Although we have restricted our discussions to the heterotic string theory and elven-dimensional supergravity, similar arguments apply to the low energy effective actions of the other string theories. We can exclude, for example, supersymmetric decay modes of type II theory compactified on a Calabi-Yau manifold, $T^2 \times K3$ or a torus by the analysis of §6 and §7.

As an aside, it is interesting to consider the implications of the dualities between theories; the heterotic string compactified to on $T^4$ is dual to the IIA string on $K3$, and the toroidal vacuum admits a decay mode by metric field charged brane formation whilst the $K3$ vacuum admits no decay modes. Now the duality relates the heterotic and IIA six dimensional fields via [38]:

$$\begin{aligned} \Phi^h = -\Phi^{II}, \qquad & G^h_{AB} = e^{-\Phi^h} G^{II}_{AB}, \\ A^h_A = A^{II}_A, \qquad & M^h = M^{II}, \\ H^h \;\; = & \;\; e^{-\Phi}\tilde{H}^{II}, \end{aligned} \tag{9.90}$$

where the heterotic metric in the string frame is $G^h_{AB}$ and the IIA metric in the string frame is $G^{II}_{AB}$; $\Phi$, $A$, $H$ are the six dimensional dilaton, 24 abelian gauge fields and antisymmetric tensor field strength respectively. $\tilde{H}$ is the (conformally invariant) dual tensor to $H$ and the $M$ fields are the matrix valued scalar field representing elements of $O(4, 20)/(O(4) \times O(20))$. Then, the solution in type II theory dual to that in the heterotic theory is:

$$ds^2_{str} \;\; = \;\; e^{\Phi^{II}} \{ \frac{dr^2}{(1 - \frac{\mu}{r^2})} + dx^2 + dy^2 + r^2 d\theta^2$$

$$
\begin{aligned}
& -r^2\cos^2\theta \mathrm{dt}^2 + \mathrm{e}^{-2\Phi^{\mathrm{II}}}(\mathrm{r}^2 - \mu)\mathrm{d}\bar{\psi}^2\} && (9.91) \\
e^{2\Phi^{II}} &= (1 - \frac{\mu}{r^2} + B^2 r^2 \sin^2\theta) && (9.92)
\end{aligned}
$$

which corresponds to a solution which is not asymptotically flat in ten dimensions, and is singular at the "horizon" $r = \sqrt{\mu}$. That is, the dualised solution does not represent an instantonic decay mode of even a non-supersymmetric vacuum of the dual theory, as we would expect.

Instabilities of vacua of supergravity theories of the type $M^4 \times K$ exist only if we include instanton solutions in which the topology is changed. Even in general relativity, the necessity of including varying topologies is not obvious, since cluster decomposition cannot be used to prove the hypothesis (unlike in Yang-Mills theory).

It has been suggested that target space duality in string theory may be used to exclude solutions of different topology [12]. In the string theory, the winding numbers about each of the compactified directions are conserved quantum numbers. As the radii of the compactified directions increase, each of the winding numbers disappears and it is impossible to change the global space topology. Since T-duality implies an equivalence relation between small and large radii of compactified directions, this suggests that decompactification instability of compactified directions is not possible either, and thus that no semiclassical instability of the vacuum exists. However, the arguments given do not apply to the Calabi-Yau vacua, since their duality groups do not generally relate small and large volume compactifications. We might expect that topology change should in any case be considered in a wider context than string (perturbation) theory.

In considering the semi-classical stability of the vacua of string theories, we have shown that instantons of different topology to the putative vacuum are excluded by the incompatibility of their asymptotic spin structure with that of the vacuum. Any instantonic decay modes must lie in a different superselection sector of the Hilbert space of states, and do not contribute to the decay rate of the supersymmetric vacuum. Although one would expect that the structure of the supersymmetry algebra at infinity would prevent the existence of even non-supersymmetric decay modes, it is reassuring that there is a natural way of excluding such instantons by semi-classical arguments.

# 10 Acknowledgements

I wish to thank Stephen Hawking for suggesting the problem and for useful discussions. I would also like to thank Gary Gibbons and Paul Townsend for helpful comments. Financial support for this work was provided by Trinity College, Cambridge.

# Appendix A Dominant energy condition

A generic energy momentum tensor $T_{MN}$ satisfies the dominant energy condition provided that for any future directed timelike or null $v$ the associated energy momentum flux $j = -T \cdot v$ satisfies the inequalities:

$$j^2 \leq 0 \text{ and } -v \cdot j \leq 0 \tag{A.1}$$

Now, for a $p$-form field $\mathcal{B}$ with associated $(p+1)$-form field strength $\mathcal{H}$, the energy momentum tensor is:

$$T_{MN} = \{g_{NN'} \mathcal{H}_{MP_1..P_p} \mathcal{H}^{N'P_1..P_p} - \frac{1}{2(1+p)} \mathcal{H}^2 g_{MN}\} \tag{A.2}$$

It is a standard algebraic exercise to show that such an energy momentum tensor satisfies the required condition, assuming that the fields are real. We refer the reader to [39] and [31] for further details.

Furthermore, (A.1) hold as strict inequalities except in special circumstances, when either $v$ or the field strengths are null; that is, the only conditions under which they hold as strict equalities are as follows:

$$\begin{aligned} j^2 = 0 \quad &\leftrightarrow \quad \mathcal{H}^2 = 0 \text{ or } v^2 = 0 \\ -j \cdot v = 0 \quad &\leftrightarrow \quad j \wedge v = 0 \text{ and } v^2 = 0 \end{aligned} \tag{A.3}$$

with the latter condition implying that $v$ must be a principle null vector of $\mathcal{H}$.

If we assume that the components of $\mathcal{H}$ are real and that $\mathcal{H}^2 < 0$, there exists a frame in which only the components $\mathcal{H}_{0P_1..P_p}$ are non zero, ie. the field is purely "electric". It is then straightforward to show that $j^2 \propto \mathcal{H}^4 v^2$ and $-j \cdot v \propto -\mathcal{H}^2 v^2$, implying $j$ satisfies the dominant energy condition. However, for a purely imaginary "electric" field, $\mathcal{H}^2 > 0$ and the energy momentum flux vector does not satisfy the dominant energy condition. This is a general statement; $\mathcal{H}$ will satisfy the dominant energy condition only if the components are real.

Since the energy momentum tensors considered in §6 and §8 are of the form (A.2) (with positive definite conformal prefactors), the dominant energy condition is satisfied provided that the fields are real on the Lorentzian section.

# References


[1] P. Candelas, G. Horowitz, A. Strominger and E. Witten, Nucl. Phys. B258 (1985) 46.

[2] M. Duff, B. Nilsson and C. Pope, Phys. Rep. 130 (1985).

[3] M. Green, J. Schwarz and E. Witten, *Superstring theory, Vol. 2*, Cambridge University Press, 1987.

[4] M. Green, J. Schwarz and P. West, Nucl. Phys. B254 (1985) 327.

[5] P. Schoen and S-T. Yau, Comm. Math. Phys. 65 (1979) 45; Phys. Rev. Lett. 43 (1979) 1457; Comm. Math. Phys. 79 (1981) 47; Comm. Math. Phys. 79 (1981) 231.

[6] E. Witten, Comm. Math. Phys. 80 (1980) 381.

[7] J. Nester, Phys. Lett. A83 (1982) 241.

[8] E. Witten, Nucl. Phys. B195 (1982) 481.

[9] D. Brill and G. Horowitz, Phys. Lett. B262 (1991) 437.

[10] P. Mazur, Nucl. Phys. B294 (1987) 525.

[11] T. Banks and L. Dixon, Nucl. Phys. B307 (1988) 93.

[12] M. Cvetic, S. Griffies and S-J. Rey, Nucl. Phys. B389 (1993) 3.

[13] F. Dowker, J. Gauntlett, G. Gibbons and G. Horowitz, Phys. Rev. D52 (1995) 6929.

[14] F. Dowker, J. Gauntlett, G. Gibbons and G. Horowitz, Phys. Rev. D53 (1996) 7115.

[15] K. Konishi, N. Maggondi and H. Panagopoulos, Nucl. Phys. B309 (1988) 201 and Nucl. Phys. B323 (1989) 441.

[16] F. Dowker, J. Gauntlett, S. Giddings and G. Horowitz, Phys. Rev. D50 (1994) 2662.

[17] G. Gibbons and S. Hawking, Comm. Math. Phys. 66 (1979) 291.

[18] T. Eguchi, P. Gilkey and A. Hanson, Phys. Rep. 66 (1980).

[19] G. Gibbons, D. Page and C. Pope, Comm. Math. Phys. (1990) 529.

[20] S. Behrndt and R. Kallosh, Nucl. Phys. B468 (1996) 85.

[21] R. Myers and M. Perry, Ann. Phys. 172 (1986) 304.

[22] A. Giveon, M. Porrati and E. Rabinovici, Phys. Rep 244 (1994).

[23] A. Sen, Int. J. Mod. Phys. A9 (1994) 3707; J. Maharana and J. Schwarz, Nucl. Phys. B390 (1993) 3.

[24] G. Gibbons and K. Maeda, Nucl. Phys. B298 (1998) 741.

[25] G. Horowitz and A. Sen, Phys. Rev. D53 (1996) 808; A. Peet, hep-th/9506200; M. Cvetic and D. Youm, hep-th/9605051.

[26] G. Gibbons and R. Kallosh, Phys. Rev. D51 (1995) 2939.

[27] A. Strominger and C. Vafa, Phys. Lett. B379 (1996) 99.

[28] S. Hawking, G. Horowitz and S. Ross, Phys. Rev. D51 (1995) 4302; S. Hawking and S. Ross, Phys. Rev. D52 (1995) 5865.

[29] G. Horowitz, hep-th/9604051.

[30] S. Hawking and G. Ellis, *The large scale structure of space-time*, Cambridge University Press, 1973.

[31] G. Horowitz, in *String theory and quantum gravity '92, ed. J. Harvey, et al.*, World Scientific, 1993; J. Harvey and A. Strominger, *ibid*; S. Kar, hep-th/9604047.

[32] L. Bombelli, R. Koul, G. Hunstatter, J. Lee and R. Sorkin, Nucl. Phys. B289 (1987) 735; S. Deser and M. Soldate, Nucl. Phys. B311 (1988) 739.

[33] G. Gibbons, S. Hawking, G. Horowitz and M. Perry, Comm. Math. Phys. 88 (1983) 295.

[34] O. Reula, J. Math. Phys. 23 (1982) 5.

[35] T. Parker and C. Taubes, Comm. Math. Phys. 84 (1982) 223.

[36] E. Cremmer, B. Julia and J. Scherk, Phys. Lett. B76 (1978) 409.

[37] E. Witten, Nucl. Phys. B443 (1995) 85; C. Hull and P. Townsend, Nucl. Phys. B438 (1995) 109; P. Townsend, Phys. Lett. B350 (1995) 184.

[38] A. Sen, Nucl. Phys. B450 (1995) 103.

[39] B. Carter, in *General relativity: an Einstein centenary survey*, Cambridge University Press, 1979.